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Global Existence, Decay, and Blowup of Solutions for Some Mildly Degenerate Nonlinear Kirchhoff Strings

Kosuke Ono*

Faculty of Integrated Arts and Sciences, The University of Tokushima, 1-1 Minamijosanjima-cho, Tokushima 770, Japan

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0. INTRODUCTION

In this paper we consider the initial-boundary value problem for the second-order hyperbolic equations

$$\begin{cases} u'' + M(\|A^{1/2}u\|^2)\, Au + \delta u' = f(u) \qquad \text{in} \quad \Omega \times [0, \infty) \\ u(x, 0) = u_0(x), \qquad u'(x, 0) = u_1(x), \qquad \text{and} \qquad u(x, t)|_{\partial\Omega} = 0, \end{cases} \tag{0.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $' = \partial_t \equiv \partial/\partial t$; $A = -\Delta \equiv -\sum_{j=1}^{N} \partial^2/\partial x_j^2$ is the Laplace operator with the domain $\mathscr{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$, $\|\cdot\|$ is the norm of $L^2(\Omega)$, $\delta \geqslant 0$, $f(u) = |u|^\alpha u$ with $\alpha > 0$, $M(r)$ is a nonnegative locally Lipschitz function for $r \geqslant 0$ like

$$M(r) = a + br^\gamma \tag{0.2}$$

with $a \geqslant 0$, $b \geqslant 0$, $a + b > 0$, and $\gamma > 0$. We call Eq. (0.1) a non-degenerate equation when $a > 0$ and $b > 0$, and a degenerate one when $a = 0$ and $b > 0$. In the case of $a > 0$ and $b = 0$, Eq. (0.1) is usual semilinear wave equations.

The case of $N = 1$, Eq. (0.1) describes the nonlinear vibrations of an elastic string. The original equation is

$$\rho h \frac{\partial^2 u}{\partial t^2} + \delta \frac{\partial u}{\partial t} = \left\{ p_0 + \frac{Eh}{2L} \int_0^L \left(\frac{\partial u}{\partial x} \right)^2 dx \right\} \frac{\partial^2 u}{\partial x^2} + f$$

for $0 < x < L$, $t \geqslant 0$, where $u = u(x, t)$ is the lateral displacement at the space coordinate x and the time t, E the Young modulus, ρ the mass density, h the cross-section area, L the length, p_0 the initial axial tension, δ the resistance modulus, and f the external force. When $\delta = f = 0$, the equation

* E-mail: ono@ias.tokushima-u.ac.jp.

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is firstly introduced by Kirchhoff [9], and is called the Kirchhoff string after his name.

When $\delta = 0$, the global in time solvability of Eq. (0.1) is rather well known in the class of analytic functions space (see [3, 16, 24]), but at the present, it is open in the class of usual Sobolev space. In such a class, only the local in time solvability has been shown by many authors (see [1, 2, 5, 25, 31, 32]). We note that in the case of unbounded domain, D'Ancona and Spagnolo [4] have shown the global existence of a unique C^∞ solution for non-degenerate equations with small C_0^∞ data. As we know well, the global in time solvability deeply depends on the decay structure of solutions to the corresponding linearized problem of (0.1). However, because the problem is given by the interior initial-boundary value problem for the nonlinear hyperbolic equations, the solutions do not have any decay properties.

On the other hand, when $\delta > 0$, utilizing a dissipative effect, we may expect certain decay properties of the solutions under suitable assumptions including $u_0 \in \mathscr{W}_*$. For example, the solutions decay at an exponential rate as $t \to \infty$ in the non-degenerate case (i.e., $a > 0$, see Section 3), and a certain algebraic rate in the degenerate case (i.e., $a = 0$, see Section 2). The degenerate case is more difficult to handle with than the non-degenerate case, and moreover, the difficulty increases in the case that the blowup term $f(u) = |u|^\alpha u$ appears because semilinear wave equations including blowup terms cause certain blowup phenomena under suitable assumptions including $u_* \in \mathscr{V}_*$ (e.g., [8, 10, 22, 23, 30]).

We define the energy and the potential associated with Eq. (0.1) by

$$E(u, u') \equiv \|u'\|^2 + J(u) \tag{0.3}$$

and

$$J(u) \equiv \left(a + \frac{b}{\gamma + 1} \|A^{1/2}u\|^{2\gamma} \right) \|A^{1/2}u\|^2 - \frac{2}{\alpha + 2} \|u\|_{\alpha+2}^{\alpha+2}, \tag{0.4}$$

respectively, where $\|\cdot\|_p$ is usual $L^p(\Omega)$-norm ($\|\cdot\| = \|\cdot\|_2$). In what follows, we denote $E(t) \equiv E(u(t), u'(t))$ and $E(0) \equiv E(u_0, u_1)$ for simplicity. We introduce the K-positive set (i.e., the modified potential well) and the K-negative set by

$$\mathscr{W}_* \equiv \{u \in \mathscr{D}(A) \colon K(u) > 0\} \cup \{0\} \tag{0.5}$$

(see Nakao and Ono [15]) and

$$\mathscr{V}_* \equiv \{u \in \mathscr{D}(A) \colon K(u) < 0\}, \tag{0.6}$$

respectively, where we set

$$K(u) \equiv \begin{cases} a\,\|A^{1/2}u\|^2 - \|u\|_{\alpha+2}^{\alpha+2} & \text{if} \quad a > 0 \\ b\,\|A^{1/2}u\|^{2(\gamma+1)} - \|u\|_{\alpha+2}^{\alpha+2} & \text{if} \quad a = 0 \end{cases} \tag{0.7}$$

(cf. [8, 23, 26] for potential well, stable set, unstable set). We note that if $u \in \mathscr{W}_*$ and $u \neq 0$, then $E(u, u') > 0$, and that if $E(u, u') < 0$, then $u \in \mathscr{V}_*$. The potential well method in [23] is used in the energy class, that is, under the restriction $\alpha \leqslant 4/(N-2)$, while the modified potential well method in [15] can be applied in the higher energy class without such restriction of α.

When $\delta > 0$ and $f(u) \equiv 0$, for degenerate equations (i.e., $a = 0$), Nishihara and Yamada [17] have proved the global existence of a unique solution under the assumptions that the initial data $\{u_0, u_1\}$ are sufficiently small and $u_0 \neq 0$ (i.e., $M(\|A^{1/2}u_0\|^2) > 0$). However, the method in [17] can not be applied directly to the case that degenerate equations have the blowup term $f(u) = |u|^\alpha u$. Our situations are more delicate and difficult. To prove the existence of global solutions for Eqs. (0.1) and (0.2) with $a = 0$, we need to derive suitable a-priori estimates including $\|Au(t)\|$ and $\|A^{1/2}u'(t)\|$ in addition to the usual energy estimate, which is the main difficulty due to the degeneracy of $M(r)$. A key point of the analysis is to show that $\|A^{1/2}u(t)\| > 0$ for all $t \geqslant 0$, because the diffusion lacks when $\|A^{1/2}u(t)\| = 0$ for some t and then we meet the derivative loss. Fortunately, by using the modified potential well method, we can apply the general theory on the energy decay of hyperbolic equations in Nakao [13], and we obtain the decay estimate of the energy $E(t)$. Then we derive the desired estimation $\|A^{1/2}u(t)\| > 0$ by help of the energy decay in Section 2. The interest of the analysis is to combine and apply the modified potential well method in [15], the energy decay estimate in [13], and the device in [17]. (See [21] for $f(u) = -|u|^\alpha u$.) But, we cannot prove that $\|A^{1/2}u(t)\| > 0$ for $t \geqslant 0$ when $N \geqslant 4$ (see Remark 2.6).

In Section 4, for non-degenerate equations (i.e., $a > 0$) we show the global existence of solutions without the restriction of the dimension N, under the assumptions that the initial energy $E(0)$ is suitably small and $u_0 \in \mathscr{W}_*$ (cf. [7] for $\gamma = 1$, $\alpha = 2$, $N = 3$). The analysis is not difficult.

On the other hand, because Eq. (0.1) has the so-called blowup term $f(u) = |u|^\alpha u$, we can show that the local solutions can not be continued globally in time under certain conditions. Then we say that the local solutions blow up at some finite time.

When the initial energy $E(0)$ is nonpositive, applying the concavity method (see Levine [10, 11] and [6]), we shall prove that the local solutions blow up at some finite time, and give upper estimates of the blowup time in Section 4. Recently, we have obtained the similar results for the problem (0.1) with a strong dissipative term Au' instead of u' in [20].

Even if the initial energy $E(0)$ is positive (but $E(0)$ is suitably small), under the restriction $\alpha \leqslant 4/(N-2)$ ($\alpha < \infty$ if $N=1, 2$) and $u_0 \in \mathscr{V}_*$, combining the so-called potential well method and the concavity method, we shall prove that the local solutions blow up at some finite time. These methods have been studied for semilinear wave equations (i.e., $b=\delta=0$) in the energy class by Payne and Sattinger [23]. (See also Ishii [8] for details). Recently, applying similar methods, Ohta [19] improved their results for nonlinear evolution equations of second order with dissipative terms ($\delta \geqslant 0$). (See [18] for another method and also [12, 22, 27]). Moreover, we shall give an upper estimate of the blowup time under $(u_0, u_1) > 0$ in Section 5.

Our plan of this paper is as follows. In Section 1, we state the local existence theorem. In Sections 2 and 3, we show the global existence and decay properties of solutions for degenerate and non-degenerate equations with a dissipative term, respectively. In Sections 4 and 5, we study the blowup problem in the cases of the initial energy being nonpositive and positive, respectively.

The notations we use in this paper are standard. We denote by H the real Hilbert space $L^2(\Omega)$. The symbol $(\cdot, \cdot)$ means the inner product in $H = L^2(\Omega)$ or sometimes duality between the space X and its dual X'. We denote by d_j, $j \geqslant 1$, various constants independent of $\{u_0, u_1\}$, and we put $[a]^+ = \max\{0, a\}$ where $1/[a]^+ = \infty$ if $[a]^+ = 0$.

1. PRELIMINARIES

By applying the Banach contraction mapping theorem, we get the following local existence theorem (see [1, 2]).

THEOREM 1.1. *Let the initial data $\{u_0, u_1\} \in \mathscr{D}(A) \times \mathscr{D}(A^{1/2})$ and $\delta \geqslant 0$, and let $M(r)$ be a nonnegative locally Lipschitz function for $r \geqslant 0$ with*

$$M(\|A^{1/2}u_0\|^2) > 0, \tag{1.1}$$

and let $f(u)$ be a nonlinear C^1-function such that

$$|f(u)| \leqslant k_1 |u|^{\alpha+1} \qquad \text{and} \qquad |f'(u)| \leqslant k_2 |u|^{\alpha} \tag{1.2}$$

with some constants k_1, k_2, and $\alpha \leqslant 2/(N-4)$ if $N \geqslant 5$. Then the problem (0.1) *admits a unique local solution u in the class*

$$C^0([0, T); \mathscr{D}(A)) \cap C^1([0, T); \mathscr{D}(A^{1/2})) \cap C^2([0, T); H) \tag{1.3}$$

for some $T = T(\|Au_0\|, \|A^{1/2}u_1\|) > 0$. *Moreover, if* $M(\|A^{1/2}u(t)\|^2) > 0$ *for* $0 \leqslant t < T$, *at least one of the following statements is valid*:

$$\text{(i)} \quad T = \infty,$$

$$\text{(ii)} \quad \|A^{1/2}u'(t)\|^2 + \|Au(t)\|^2 \to \infty \qquad as \quad t \to T^-,$$

$$\text{(iii)} \quad M(\|A^{1/2}u(t)\|^2) \to 0 \qquad as \quad t \to T^-.$$

Remark 1.2. When $M(r) = r^\gamma \in C^1([0, \infty))$, we assume that $u_0 \neq 0$ for the degeneracy condition (1.1) of $M(r)$.

Proof. For $T > 0$ and $R > 0$, we define the two-parameter space of the solutions as

$$X_{T, R} \equiv \{v(t) \in C^0_w([0, T]; \mathscr{D}(A)) \cap C^1_w([0, T]; \mathscr{D}(A^{1/2})) \colon \|A^{1/2}v'(t)\|^2 + \|Av(t)\|^2 \leqslant R^2 \text{ on } [0, T],\ v(0) = u_0,\ v'(0) = u_1\}.$$

It is easy to see that $X_{T, R}$ can be organized as a complete metric space with the distance

$$d(u, v) \equiv \sup_{0 \leqslant t \leqslant T} \{\|u'(t) - v'(t)\|^2 + \|A^{1/2}(u(t) - v(t))\|^2\}.$$

We define the nonlinear mapping $\mathscr{S}$ as follows. For $v \in X_{T, R}$, $u = \mathscr{S}v$ is the unique solution of the linear equations

$$\begin{cases} u'' + M(\|A^{1/2}v\|^2)\, Au + \delta u' = f(v) & \text{in} \quad \Omega \times [0, T] \\ u(0) = u_0, \qquad u'(0) = u_1, \qquad \text{and} & u|_{\partial\Omega} = 0. \end{cases}$$

Using the fact that $M(\|A^{1/2}u_0\|^2) \equiv M_0 > 0$ (i.e., (1.1)), we prove that there exist $T > 0$ and $R > 0$ such that $\mathscr{S}$ maps $X_{T, R}$ into itself; $\mathscr{S}$ is a contraction mapping with respect to the metric $d(\cdot, \cdot)$. By applying the Banach contraction mapping theorem, we obtain a unique solution u belonging to $X_{T, R}$ of (0.1). Therefore, it follows from the continuity argument for wave equations (e.g., [28, 29]) that this solution u belongs to (1.3). We omit the detail here. (See Ono [21] for details.) ∎

We use the following well-known lemma without the proof in this paper.

LEMMA 1.3. (Gagliardo–Nirenberg). *Let* $1 \leqslant r < p \leqslant \infty$ *and* $p \geqslant 2$. *Then, the inequality*

$$\|v\|_p \leqslant c_* \|A^{m/2}v\|^\theta \|v\|_r^{1-\theta} \qquad \textit{for} \quad v \in \mathscr{D}(A^{m/2}) \cap L^r(\Omega)$$

holds with some constant c_* *and*

$$\theta = \left(\frac{1}{r} - \frac{1}{p}\right)\left(\frac{1}{r} + \frac{m}{N} - \frac{1}{2}\right)^{-1}$$

provided that $0 < \theta \leqslant 1$ ($0 < \theta < 1$ *if* $m - N/2$ *is a nonnegative integer*).

(Sobolev–Poincaré) *Let* $1 \leqslant p \leqslant 2N/[N-2m]^+$ ($1 \leqslant p < \infty$ *if* $N = 2m$). *Then, the inequality*

$$\|v\|_p \leqslant c_* \|A^{m/2}v\| \qquad \text{for} \quad v \in \mathscr{D}(A^{m/2}))$$

holds with some constant c_*.

2. MILDLY DEGENERATE CASE

In this section we shall consider the global existence and decay properties of the solution for the following degenerate nonlinear wave equations with a dissipative term ($M(r) = r^\gamma \in C^1([0, \infty))$ and $\delta = 1$ in (0.1) for simplicity),

$$\begin{cases} u'' + \|A^{1/2}u\|^{2\gamma} Au + u' = |u|^\alpha u & \text{in} \quad \Omega \times [0, \infty) \\ u(0) = u_0, \qquad u'(0) = u_1, \qquad \text{and} & u|_{\partial\Omega} = 0, \end{cases} \tag{2.1}$$

where $\gamma \geqslant 1$ and $\alpha > 0$. We note that the problem (2.1) has the trivial solution $u \equiv 0$ (then $E(u, u') = 0$). Our purpose is to seek non-trivial global solutions. We recall the energy, the potential, and the K-positive set associated with Eq. (2.1),

$$E(u, u') \equiv \|u'\|^2 + J(u), \tag{2.2}$$

$$J(u) \equiv \frac{1}{\gamma+1} \|A^{1/2}u\|^{2(\gamma+1)} - \frac{2}{\alpha+2} \|u\|_{\alpha+2}^{\alpha+2}, \tag{2.3}$$

and

$$\mathscr{W}_* \equiv \{u \in \mathscr{D}(A)\colon K(u) \equiv \|A^{1/2}u\|^{2(\gamma+1)} - \|u\|_{\alpha+2}^{\alpha+2} > 0\} \cup \{0\}, \tag{2.4}$$

respectively. Then we observe the following.

PROPOSITION 2.1. (i) *If* $\alpha > 2\gamma$ *and* $\alpha \leqslant 4/(N-2)$ ($\alpha < \infty$ *if* $N = 1, 2$), *or if* $(4-N)\,\alpha + 4 > 2\gamma$ *and* $\alpha > 4/[N-2]^+$, *then*

$$\mathscr{W}_* \text{ is a neighborhood of } 0 \text{ in } \mathscr{D}(A^{1/2}) = H_0^1(\Omega) \text{ and an open set.}$$

(ii) *If* $u \in \overline{\mathscr{W}_*}$ *and* $\alpha > 2\gamma$, *then*

$$d_*^{-1} \|A^{1/2}u\|^{2(\gamma+1)} \leqslant J(u) \qquad (\leqslant E(u, u')) \tag{2.5}$$

with $d_* = (\alpha+2)(\gamma+1)/(\alpha-2\gamma)$.

Proof. We have from Lemma 1.3 that

$$\|u\|_{\alpha+2}^{\alpha+2} \leqslant c_*^{\alpha+2} \|A^{1/2}u\|^{(\alpha-2\gamma)-(\alpha+2)\theta_1} \|Au\|^{(\alpha+2)\theta_1} \|A^{1/2}u\|^{2(\gamma+1)} \tag{2.6}$$

with $\theta_1 = [(N-2)\alpha - 4]^+/(2(\alpha+2))$ under $(\alpha-2\gamma)-(\alpha+2)\theta_1 > 0$, and hence, we see that $K(u) > 0$ if the $\mathscr{D}(A^{1/2})$-norm of u is sufficiently small and $u \neq 0$, which implies (i). By the definitions of $J(u)$ and $\mathscr{W}_*$, (2.5) holds true. ∎

In order to derive the decay estimate of the energy $E(t)$, we use the following useful lemma.

Lemma 2.2. (Nakao [14]). *Let* ϕ *be a non-increasing nonnegative function on* $[0, \infty)$ *satisfying*

$$\phi(t)^{1+r} \leqslant k\{\phi(t) - \phi(t+1)\} \tag{2.7}$$

for $r > 0$ *and* $k > 0$. *Then*

$$\phi(t) \leqslant \{\phi(0)^{-r} + rk^{-1}[t-1]^+\}^{-1/r} \qquad \textit{for} \quad t \geqslant 0. \tag{2.8}$$

Proof. Setting $\psi(t) = \phi(t)^{-r}$, we see from (2.7) that

$$\begin{aligned}\psi(t+1) - \psi(t) &= \int_0^1 \frac{d}{d\theta} \{\theta\phi(t+1) + (1-\theta)\,\phi(t)\}^{-r}\, d\theta \\ &= r\int_0^1 \{\theta\phi(t+1) + (1-\theta)\,\phi(t)\}^{-r-1}\, d\theta\{\phi(t) - \phi(t+1)\} \\ &\geqslant rk^{-1}.\end{aligned}$$

Then, we get

$$\psi(t+1) \geqslant \psi(0) + rk^{-1}t$$

and the desired estimate (2.8). ∎

Using above lemma, we obtain the following energy decay estimate.

PROPOSITION 2.3. *Let u be a solution of* (2.1) *on* $[0, T]$. *If* $u \in \overline{\mathscr{W}_*}$, $\alpha > 2\gamma$, *and*

$$K(u) \geqslant (1/2)\ \|A^{1/2}u\|^{2(\gamma+1)}, \tag{2.9}$$

then

$$E(t) \leqslant \{E(0)^{-\gamma/(\gamma+1)} + d_0^{-1}[t-1]^+\}^{-(\gamma+1)/\gamma} \tag{2.10}$$

for $0 \leqslant t \leqslant T$, *where* d_0 *is some positive constant depending only on* γ, d_*, c_* *if* $E(0) \leqslant 1$.

Proof. Following Nakao [13], we shall prove the theorem. Multiplying Eq. (2.1) by $2u'$ and integrating it over Ω, we have

$$E'(t) + 2\ \|u'(t)\|^2 = 0, \tag{2.11}$$

and hence, $E(t)$ is a non-increasing function. Moreover, from (2.5), $E(t)$ is nonnegative. Integrating (2.11) over $[0, t]$, we have that

$$E(t) + 2 \int_0^t \|u'(s)\|^2\, ds = E(0). \tag{2.12}$$

For a moment, we assume that $T > 1$. Integrating (2.11) over $[t, t+1]$, $0 < t < T-1$, we have

$$2 \int_t^{t+1} \|u'(s)\|^2\, ds = E(t) - E(t+1) \qquad (\equiv 2D(t)^2). \tag{2.13}$$

Then, there exist two numbers $t_1 \in [t, t+1/4]$ and $t_2 \in [t+3/4, t+1]$ such that

$$\|u'(t_i)\| \leqslant 2D(t) \qquad \text{for} \quad i = 1, 2. \tag{2.14}$$

Multiplying Eq. (2.1) by u and integrating it over Ω, we have

$$K(u(t)) = \|u'(t)\|^2 - (u(t), u'(t)) - \partial_t(u(t), u'(t)),$$

and integrating the resulting equality over $[t_1, t_2]$, we observe from (2.9) that

$$
\begin{aligned}
(1/2) \int_{t_1}^{t_2} \|A^{1/2}u(s)\|^{2(\gamma+1)}\, ds
&\leqslant \int_{t_1}^{t_2} K(u(s))\, ds \\
&\leqslant \int_t^{t+1} \|u'(s)\|^2\, ds + \left\{ \left(\int_t^{t+1} \|u'(s)\|^2\, ds \right)^{1/2} + \sum_{i=1}^{2} \|u'(t_i)\| \right\} \sup_{t \leqslant s \leqslant t+1} \|u(s)\| \\
&\leqslant D(t)^2 + 5c_* D(t)(d_* E(t))^{1/(2(\gamma+1))},
\end{aligned} \tag{2.15}
$$

where we used (2.13), (2.14), and Lemma 1.3 at the last inequality.

Integrating (2.11) over $[t, t_2]$, we have from (2.13) and (2.15) that

$$
\begin{aligned}
E(t) &= E(t_2) + 2 \int_t^{t_2} \|u'(s)\|^2\, ds \\
&\leqslant 2 \int_{t_1}^{t_2} E(s)\, ds + 2 \int_t^{t+1} \|u'(s)\|^2\, ds \\
&\leqslant 4 \int_t^{t+1} \|u'(s)\|^2\, ds + \int_{t_1}^{t_2} \|A^{1/2}u(s)\|^{2(\gamma+1)}\, ds \\
&\leqslant 6D(t)^2 + 10c_* D(t)(d_* E(t))^{1/(2(\gamma+1))} \\
&\leqslant 6D(t)^2 + d_*^2 (10c_* D(t))^{2(\gamma+1)/(2\gamma+1)} + (1/2)\, E(t).
\end{aligned}
$$

Noting the fact that $2D(t)^2 \leqslant E(t) \leqslant E(0)$ (see (2.12), (2.13)), we have

$$
E(t) \leqslant c_0\, D(t)^{2(\gamma+1)/(2\gamma+1)}
$$

with $c_0 = 2\{6(E(0)/2)^{\gamma/(2\gamma+1)} + d_*^2 (10c_*)^{2(\gamma+1)/(2\gamma+1)}\}$, and hence, from (2.13) we see

$$
E(t)^{1+\gamma/(\gamma+1)} \leqslant 2^{-1} c_0^{(2\gamma+1)/(\gamma+1)} \{E(t) - E(t+1)\}.
$$

Noting (2.12) and applying the Lemma 2.2, we obtain the desired estimate (2.10). ∎

To state our result, we introduce a function $H(t)$ as

$$
H(t) \equiv \frac{\|A^{1/2}u'(t)\|^2}{\|A^{1/2}u(t)\|^{2\gamma}} + \|Au(t)\|^2 \tag{2.16}
$$

for $t \geqslant 0$ (see Nishihara and Yamada [17]). Our main result is as follows.

THEOREM 2.4. *Let* $1 \leqslant N \leqslant 3$. *Suppose that*

$$\alpha > 2\gamma \qquad \text{if} \quad N = 1, 2, \tag{2.17}$$

$$\alpha > \max\{4\gamma - 2, 2\gamma + [\alpha - 4]^{+}/2\} \qquad \text{if} \quad N = 3, \tag{2.18}$$

and that the initial data $\{u_0, u_1\}$ *belong to* $\mathscr{W}_{*} \times \mathscr{D}(A^{1/2})$ *with*

$$u_0 \neq 0 \tag{2.19}$$

and are suitably small, that is, $\{u_0, u_1\}$ *satisfy*

$$(0<) \quad d_1 E(0)^{(2(\alpha - 2\gamma) - [(N-2)\alpha - 4]^{+})/(4(\gamma + 1))} < 1 \tag{2.20}$$

for $1 \leqslant N \leqslant 3$ *and*

$$(0<) \quad d_2^2 E(0)^{(\gamma - 1)/(\gamma + 1)} \{ H(0)^{1 - (N-1)\varepsilon} + d_3 E(0)^{(\alpha + 1 - (N-1)\varepsilon - 2\gamma)/(\gamma + 1)} \}^{1/(1 - (N-1)\varepsilon)} < 1 \tag{2.21}$$

for $0 < \varepsilon < 1$ *if* $N = 1, 2$ *or*

$$(0<) \quad \{ (d_2 E(0)^{(\gamma - 1)/(2(\gamma + 1))})^{\alpha - 2} + d_4 E(0)^{(\alpha + 2 - 4\gamma)/(2(\gamma + 1))} \}^{2/(\alpha - 2)} H(0) < 1 \tag{2.22}$$

if $N = 3$, *where* d_1, d_2, d_3, d_4 *are certain positive constants depending only on* α, γ, c_{*}. *Then the problem* (2.1) *admits a unique global solution* $u \in \mathscr{W}_{*}$ *in the class*

$$C^0([0, \infty); \mathscr{D}(A)) \cap C^1([0, \infty); \mathscr{D}(A^{1/2})) \cap C^2([0, \infty); H). \tag{2.23}$$

Moreover, this solution u satisfies

$$0 < \|A^{1/2} u(t)\|^2 \leqslant C(1 + t)^{-1/\gamma}, \tag{2.24}$$

$$\|u'(t)\|^2 + \|u''(t)\|^2 \leqslant C(1 + t)^{-1 - 1/\gamma}, \tag{2.25}$$

$$\|A^{1/2} u'(t)\|^2 \leqslant C(1 + t)^{-1} \tag{2.26}$$

with some constant C *for* $t \geqslant 0$.

Remark 2.5. The conclusion of Theorem 2 holds for the problem (0.1) with $M(r) = br^{\gamma}$, $\delta > 0$, and the nonlinear term $f(u)$ satisfying (1.2). Then we need to redefine $J(u)$ and $K(u)$ as

$$J(u) \equiv \frac{b}{\gamma + 1} \|A^{1/2} u\|^{2(\gamma + 1)} - 2 \int_{\Omega} F(u)\, dx, \quad F(u) \equiv \int_0^u f(\eta)\, d\eta,$$

and

$$K(u) \equiv b\ \|A^{1/2}u\|^{2(\gamma+1)} - k_1\ \|u\|_{\alpha+2}^{\alpha+2}$$

Remark 2.6. For general N, we need the relations $\gamma \geqslant 1$ and $\alpha(1-\theta_2) > 2\gamma$ with $\theta_2 = [(N-2)\alpha - 2]^+/(2\alpha)$ instead of (2.17) and (2.18), but such relations can not be satisfied when $N \geqslant 4$. In the case of $N \geqslant 4$, we do not have any results.

Proof. Since $u_0 \in \mathscr{W}_*$ and $\mathscr{W}_*$ is an open set, putting

$$T_1 \equiv \sup\{t \in [0, \infty): u(s) \in \mathscr{W}_* \text{ for } 0 \leqslant s < t\}$$

we see that $T_1 > 0$ and $u(t) \in \mathscr{W}_*$ for $0 \leqslant t < T_1$. If $T_1 < \infty$, then $u(T_1) \in \partial\mathscr{W}_*$, that is, one of the following two cases happens:

$$K(u(T_1)) = 0 \qquad \text{and} \qquad u(T_1) \neq 0, \tag{2.27}$$

or (by $u(T_1) \notin \mathscr{D}(A)$)

$$K(u(t)) \to \infty \qquad \text{as} \quad t \to T^-. \tag{2.28}$$

We see from (2.5), (2.6), (2.12), (2.16) that

$$\|u(t)\|_{\alpha+2}^{\alpha+2} \leqslant (1/2)\ G(t)\ \|A^{1/2}u(t)\|^{2(\gamma+1)} \tag{2.29}$$

for $0 \leqslant t < T_1$, where we set

$$G(t) \equiv d_1\, E(0)^{((\alpha-2\gamma)-(\alpha+2)\,\theta_1)/(2(\gamma+1))}\ H(t)^{(\alpha+2)\,\theta_1/2} \tag{2.30}$$

with $d_1 = 2c_*^{\alpha+2}\, d_*^{((\alpha-2\gamma)-(\alpha+2)\,\theta_1)/(2(\gamma+1))}$ and $\theta_1 = [(N-2)\alpha - 4]^+/(2(\alpha+2))$ under $(\alpha - 2\gamma) > (\alpha+2)\,\theta_1$ (i.e., (2.17), (2.18)).

Since $G(0) < 1$ for small initial data (i.e., (2.20)), putting

$$T_2 \equiv \sup\{t \in [0, \infty): G(s) < 1 \text{ for } 0 \leqslant s < t\}$$

we see that $T_2 > 0$ and $G(t) < 1$ for $0 \leqslant t < T_2$. If $T_2 < T_1$ $(< \infty)$, then we see

$$G(T_2) = 1, \tag{2.31}$$

and from (2.4) and (2.29) we have

$$\begin{aligned} K(u(t)) &\geqslant \|A^{1/2}u(t)\|^{2(\gamma+1)} - (1/2)\ G(t)\ \|A^{1/2}u(t)\|^{2(\gamma+1)} \\ &\geqslant (1/2)\ \|A^{1/2}u(t)\|^{2(\gamma+1)} \end{aligned} \tag{2.32}$$

for $0 \leqslant t \leqslant T_2$.

Since $\|A^{1/2}u_0\| > 0$ by $u_0 \neq 0$, putting

$$T_3 \equiv \sup\{t \in [0, \infty) \colon \|A^{1/2}u(s)\| > 0 \text{ for } 0 \leqslant s < t\},$$

we see that $T_3 > 0$ and $\|A^{1/2}u(t)\| > 0$ for $0 \leqslant t < T_3$. If $T_3 < T_2$, then we see

$$\|A^{1/2}u(T_3)\| = 0. \tag{2.33}$$

Multiplying Eq. (2.1) by $2Au'$ and integrating it over Ω, we have

$$\partial_t \|A^{1/2}u'(t)\|^2 + \|A^{1/2}u(t)\|^{2\gamma}\, \partial_t \|Au(t)\|^2 + 2\,\|A^{1/2}u'(t)\|^2$$
$$= 2(A^{1/2}f(u(t)), A^{1/2}u'(t)).$$

Moreover, multiplying the resulting equality by $\|A^{1/2}u(t)\|^{-2\gamma}$ for $0 \leqslant t < T_3$, we get

$$\begin{aligned} H'(t) + 2\,\frac{\|A^{1/2}u'(t)\|^2}{\|A^{1/2}u(t)\|^{2\gamma}} &= 2\gamma\,\frac{(A^{1/2}u(t), A^{1/2}u'(t))\,\|A^{1/2}u'(t)\|^2}{\|A^{1/2}u(t)\|^{2(\gamma+1)}} \\ &\quad + 2\,\frac{(A^{1/2}f(u(t)), A^{1/2}u'(t))}{\|A^{1/2}u(t)\|^{2\gamma}} \\ &\equiv I_1(t) + I_2(t), \end{aligned} \tag{2.34}$$

where $H(t)$ is given by (2.16). We observe from (2.5) and (2.12) that

$$\begin{aligned} I_1(t) &\leqslant 2\gamma\,\|A^{1/2}u(t)\|^{\gamma-1}\,\frac{\|A^{1/2}u'(t)\|^3}{\|A^{1/2}u(t)\|^{3\gamma}} \\ &\leqslant 2\gamma(d_* E(0))^{(\gamma-1)/(2(\gamma+1))}\, H(t)^{1/2}\,\frac{\|A^{1/2}u'(t)\|^2}{\|A^{1/2}u(t)\|^{2\gamma}}. \end{aligned}$$

On the other hand, we see from Lemma 1.3 that

$$\begin{aligned} |(A^{1/2}f(u), A^{1/2}u')| &\leqslant (\alpha+1)\,\|u\|_\infty^\alpha\,\|A^{1/2}u\|\,\|A^{1/2}u'\| \\ &\leqslant c_*^\alpha(\alpha+1)\,\|A^{1/2}u\|^{\alpha+1-(N-1)\varepsilon}\,\|Au\|^{(N-1)\varepsilon}\,\|A^{1/2}u'\| \end{aligned}$$

for $0 < \varepsilon < 1$ if $N = 1, 2$ and

$$\begin{aligned} |(A^{1/2}f(u), A^{1/2}u')| &\leqslant (\alpha+1)\,\|u\|_{3\alpha}^\alpha\,\|A^{1/2}u\|_6\|A^{1/2}u'\| \\ &\leqslant c_*^{\alpha+1}(\alpha+1)\|A^{1/2}u\|^{\alpha/2+1}\,\|Au\|^{\alpha/2}\,\|A^{1/2}u'\| \end{aligned}$$

if $N=3$, and hence,

$$I_2(t) \leqslant \{c_*^{\alpha}(\alpha+1)\ \|A^{1/2}u\|^{\alpha+1-(N-1)\varepsilon-\gamma}\}^2\ H(t)^{(N-1)\varepsilon} + \frac{\|A^{1/2}u'(t)\|^2}{\|A^{1/2}u(t)\|^{2\gamma}}$$

if $N=1$, 2 and

$$I_2(t) \leqslant \{c_*^{\alpha+1}(\alpha+1)\ \|A^{1/2}u(t)\|^{\alpha/2+1-\gamma}\}^2\ H(t)^{\alpha/2} + \frac{\|A^{1/2}u'(t)\|^2}{\|A^{1/2}u(t)\|^{2\gamma}}$$

if $N=3$.

Thus we obtain from (2.34) that

$$H'(t) + [1-F(t)]\,\frac{\|A^{1/2}u'(t)\|^2}{\|A^{1/2}u(t)\|^{2\gamma}} \leqslant \begin{cases} \{c_*^{\alpha}(\alpha+1)\ \|A^{1/2}u\|^{\alpha+1-(N-1)\varepsilon-\gamma}\}^2\ H(t)^{(N-1)\varepsilon} \\ \quad \text{if} \quad N=1, 2 \\ \{c_*^{\alpha+1}(\alpha+1)\ \|A^{1/2}u(t)\|^{\alpha/2+1-\gamma}\}^2\ H(t)^{\alpha/2} \\ \quad \text{if} \quad N=3, \end{cases} \tag{2.35}$$

where we set

$$F(t) \equiv d_2 E(0)^{(\gamma-1)/(2(\gamma+1))}\ H(t)^{1/2} \qquad \text{with} \quad d_2 = 2\gamma\, d_*. \tag{2.36}$$

Since $F(0)<1$ for small initial data (i.e., (2.20)–(2.22)), putting

$$T_4 \equiv \sup\{t \in [0, \infty): F(s) < 1 \text{ for } 0 \leqslant s < t\},$$

we see that $T_4 > 0$ and $F(t) < 1$ for $0 \leqslant t < T_4$. If $T_4 < T_3$, then we see

$$F(T_4) = 1. \tag{2.37}$$

Moreover, we have from (2.35) that

$$H(t) \leqslant \left\{ H(0)^{1-(N-1)\varepsilon} + c_1 \int_0^t \|A^{1/2}u(s)\|^{2(\alpha+1-(N-1)\varepsilon-\gamma)}\, ds \right\}^{1/(1-(N-1)\varepsilon)} \tag{2.38}$$

with $c_1 = c_*^{2\alpha}(1-(N-1)\varepsilon)(\alpha+1)^2$ if $N=1$, 2 and

$$H(t) \leqslant \left\{ H(0)^{-(\alpha-2)/2} - c_2 \int_0^t \|A^{1/2}u(s)\|^{\alpha+2-2\gamma}\, ds \right\}^{-2/(\alpha-2)} \tag{2.39}$$

with $c_2 = c_*^{2(\alpha+1)}(\alpha/2-1)(\alpha+1)^2$ if $N=3$.

We observe from (2.6) and (2.10) that

$$
\begin{aligned}
\int_0^t \|A^{1/2}u(s)\|^{\omega}\, ds
&\leqslant \int_0^t (d_* E(s))^{\omega/(2(\gamma+1))}\, ds \\
&\leqslant \int_0^t [d_*\{E(0)^{-\gamma/(\gamma+1)} + d_0^{-1}[s-1]^+\}^{-(\gamma+1)/\gamma}]^{\omega/(2(\gamma+1))}\, ds \\
&\leqslant 2d_*^{\omega/(2(\gamma+1))}\, d_0 E(0)^{(\omega-2\gamma)/(2(\gamma+1))}
\end{aligned}
\tag{2.40}
$$

under $\omega > 2\gamma$ and $E(0) \leqslant 1$.

Since the conditions (2.21) and (2.22) infer

$$
d_2 E(0)^{(\gamma-1)/(2(\gamma+1))} \times \{H(0)^{1-(N-1)\varepsilon} + d_3 E(0)^{(\alpha+1-(N-1)\varepsilon-2\gamma)/(\gamma+1)}\}^{1/(2(1-(N-1)\varepsilon))} < 1
$$

with $d_3 = 2c_1\, d_*^{(\alpha+1-(N-1)\varepsilon-\gamma)/(\gamma+1)}\, d_0$ under $\alpha > 2\gamma + (N-1)\varepsilon - 1$ if $N =$ 1, 2 and

$$
d_2 E(0)^{(\gamma-1)/(2(\gamma+1))}\{H(0)^{-(\alpha-2)/2} - d_4 E(0)^{(\alpha+2-4\gamma)/(2(\gamma+1))}\}^{-1/(\alpha-2)} < 1
$$

with $d_4 = 2c_2\, d_*^{(\alpha+2-2\gamma)/(2(\gamma+1))}\, d_0$ under $\alpha > 4\gamma - 2$ if $N = 3$, respectively, we get that

$$
F(t) \equiv d_2 E(0)^{(\gamma-1)/(2(\gamma+1))}\, H(t)^{1/2} < 1
$$

for $1 \leqslant N \leqslant 3$ and $0 \leqslant t \leqslant T_4$, which contradicts (2.37), and hence, we see $T_4 \geqslant T_3$. Then we have

$$
H(t) \equiv \frac{\|A^{1/2}u'(t)\|^2}{\|A^{1/2}u(t)\|^{2\gamma}} + \|Au(t)\|^2 \leqslant (d_2 E(0)^{(\gamma-1)/(2(\gamma+1))})^{-2} \tag{2.41}
$$

for $0 \leqslant t < T_3$. In particular, when $N = 3$, we also see from (2.22), (2.39), (2.40) that

$$
H(t) \leqslant 1 \qquad \text{for} \quad 0 \leqslant t < T_3. \tag{2.42}
$$

Next, we shall show that $\|A^{1/2}u(t)\| > 0$ for $t \geqslant 0$. Since $\|A^{1/2}u(T_3)\| = 0$ by (2.33), we see from (2.41) and the continuity that $\|A^{1/2}u'(T_3)\|$ must be zero. We perform the change of variable $t/T_3 - t$, then $\tilde{u}(t) = u(T_3 - t)$ satisfies

$$\begin{cases} \tilde{u}'' + \|A^{1/2}\tilde{u}\|^{2\gamma} A\tilde{u} = \tilde{u}' + f(\tilde{u}) & \text{in} \quad \Omega \times [0, T_3] \\ \tilde{u}(0) = \tilde{u}'(0) = 0 \quad \text{and} & \tilde{u}|_{\partial\Omega} = 0. \end{cases}$$

Multiplying this equation by $2\tilde{u}'$ as in (2.11) and integrating it over Ω, we have from (2.2) and (2.5) that

$$\partial_t E(\tilde{u}(t), \tilde{u}'(t)) = 2\,\|\tilde{u}'(t)\|^2 \leqslant 2\{\|\tilde{u}'(t)\|^2 + J(\tilde{u}(t))\} = 2E(\tilde{u}(t), \tilde{u}'(t)),$$

and hence,

$$E(\tilde{u}(t), \tilde{u}'(t)) \leqslant 2 \int_0^t E(\tilde{u}(s), \tilde{u}'(s))\, ds$$

for $0 \leqslant t \leqslant T_3$. Noting the fact $E(\tilde{u}(0), \tilde{u}'(0)) = 0$ and applying the Gronwall inequality, we have that

$$d_*^{-1}\,\|A^{1/2}\tilde{u}(t)\|^{2(\gamma+1)} \leqslant E(\tilde{u}(t), \tilde{u}'(t)) = 0,$$

that is, $\|A^{1/2}u(T_3 - t)\| = 0$ for $0 \leqslant t \leqslant T_3$, which contradicts the condition $u_0 \neq 0$, and hence, we see $T_3 \geqslant T_2$.

Moreover, noting the facts that $\theta_1 = [(N-2)\,\alpha - 4]^+/(2(\alpha+2)) = 0$ if $N = 1$, 2 and $H(t) \leqslant 1$ if $N = 3$ (see (2.42)), we observe from (2.20) and (2.30) that under $\alpha > 2\gamma + (\alpha+2)\,\theta_1$,

$$G(t) \leqslant d_1\, E(0)^{((\alpha - 2\gamma) - (\alpha+2)\,\theta_1)/(2(\gamma+1))} < 1$$

for $0 \leqslant t \leqslant T_2$, which contradicts (2.31), and hence, we see $T_2 \geqslant T_1$. Then, since $\|Au(t)\| \leqslant C < \infty$ for $0 \leqslant t < T_1$, we have

$$K(u(t)) \leqslant \|A^{1/2}u(t)\|^{2(\gamma+1)} \leqslant C < \infty$$

for $0 \leqslant t < T_1$, and hence, (2.28) does not happen. On the other hand, if (2.27) happens, the inequalities (2.29) and (2.32) are valid for $0 \leqslant t \leqslant T_1$ and we observe from (2.27) and (2.32) that

$$0 = K(u(T_1)) \geqslant (1/2)\,\|A^{1/2}u(T_1)\|^{2(\gamma+1)} > 0,$$

which is a contradiction, and hence, we see $T_1 = \infty$. Thus, we get $\|A^{1/2}u(t)\| > 0$ for $t \geqslant 0$. Moreover, (2.10) and (2.41) hold for $t \geqslant 0$ and the local solution u of Eq. (2.1) in the sense of Theorem 1.1 can be continued globally in time.

Lastly, we shall derive decay estimates of $\|A^{1/2}u'(t)\|$ and $\|u''(t)\|$. It follows from (2.5), (2.10), (2.41) that

$$\|A^{1/2}u'(t)\|^2 \leqslant (d_2 E(0)^{(\gamma-1)/(2(\gamma+1))})^{-2} \|A^{1/2}u(t)\|^{2\gamma} \leqslant C(1+t)^{-1}.$$

Since u is a solution of Eq. (2.1), we see

$$\begin{aligned} \|u''(t)\|^2 &\leqslant \{ \|A^{1/2}u(t)\|^{2\gamma} \|Au(t)\| + \|u'(t)\| \\ &\qquad + c_*^{\alpha+1} \|A^{1/2}u(t)\|^{(\alpha+4)/2} \|Au(t)\|^{(\alpha-2)/2} \}^2 \\ &\leqslant C(1+t)^{-\omega}, \end{aligned} \tag{2.43}$$

where $\omega = \min\{2, 1+1/\gamma, (\alpha+4)/(2\gamma)\} = 1+1/\gamma$. The proof of Theorem 2.4 is now completed. ∎

3. NON-DEGENERATE CASE

In this section we shall consider the global existence and decay properties of the solution for the following non-degenerate nonlinear wave equations with a dissipative term ($M(r) = 1 + br^\gamma \in C^1([0, \infty))$ and $\delta = 1$ in (0.1) for simplicity),

$$\begin{cases} u'' + (1 + b\|A^{1/2}u\|^{2\gamma}) Au + u' = |u|^\alpha u & \text{in} \quad \Omega \times [0, \infty) \\ u(0) = u_0, \qquad u'(0) = u_1, \qquad \text{and} \qquad u|_{\partial\Omega} = 0, \end{cases} \tag{3.1}$$

where $b \geqslant 0$, $\gamma \geqslant 1$, and $\alpha > 0$. We recall the energy, the potential, and the K-positive set associated with Eq. (3.1),

$$E(u, u') \equiv \|u'\| + J(u), \tag{3.2}$$

$$J(u) \equiv \left(1 + \frac{b}{\gamma+1} \|A^{1/2}u\|^{2\gamma}\right) \|A^{1/2}u\|^2 - \frac{2}{\alpha+2} \|u\|_{\alpha+2}^{\alpha+2}, \tag{3.3}$$

and

$$\mathscr{W}_* \equiv \{u \in \mathscr{D}(A) : K(u) \equiv \|A^{1/2}u\|^2 - \|u\|_{\alpha+2}^{\alpha+2} > 0\} \cup \{0\}, \tag{3.4}$$

respectively. We define the second energy by

$$E_2(u, u') \equiv \|A^{1/2}u'\|^2 + (1 + b\|A^{1/2}u\|^{2\gamma}) \|Au\|^2, \tag{3.5}$$

where we denote $E_2(t) \equiv E_2(u(t), u'(t))$ and $E_2(0) \equiv E_2(u_0, u_1)$ for simplicity.

Immediately, we observe the following.

PROPOSITION 3.1. (i) *If* $\alpha < 4/[N-4]^+$, *then*

$$\mathscr{W}_* \text{ is a neighborhood of } 0 \text{ in } \mathscr{D}(A^{1/2}) \text{ and an open set.}$$

(ii) *If* $u \in \overline{\mathscr{W}_*}$, *then*

$$\frac{\alpha}{\alpha+2} \|A^{1/2}u\|^2 \leqslant J(u) \qquad (\leqslant E(u, u')). \tag{3.6}$$

Proof. We have from Lemma 1.3 that

$$\|u\|_{\alpha+2}^{\alpha+2} \leqslant c_*^{\alpha+2} \|A^{1/2}u\|^{\alpha-(\alpha+2)\theta_1} \|Au\|^{(\alpha+2)\theta_1} \|A^{1/2}u\|^2 \tag{3.7}$$

with $\theta_1 = [(N-2)\alpha - 4]^+/(2(\alpha+2))$ under $\alpha - (\alpha+2)\theta_1 > 0$, and hence, we see that $K(u) > 0$ if the $\mathscr{D}(A^{1/2})$-norm of u is sufficiently small and $u \neq 0$, which implies (i). By the definitions of $J(u)$ and $\mathscr{W}_*$, (3.6) holds true. ∎

Our result is as follows.

THEOREM 3.2. *Let the initial data* $\{u_0, u_1\}$ *belong to* $\mathscr{W}_* \times \mathscr{D}(A^{1/2})$. *Suppose that*

$$\alpha < 2/[N-4]^+ \tag{3.8}$$

and that the initial energy $E(0)$ *is suitably small* (*but we can take* $E_2(0) \geqslant 1$), *that is,*

$$\max\{d_5 E(0)^{(\alpha-(\alpha+2)\theta_1)/2} E_2(0)^{(\alpha+2)\theta_1/2},\\ d_6 b E(0)^{\gamma-1/2} E_2(0)^{1/2} + d_7 E(0)^{\alpha(1-\theta_2)/2} E_2(0)^{\alpha\theta_2/2}\} < 1, \tag{3.9}$$

where d_5, d_6, d_7 *are certain positive constants depending only on* α, γ, c_*, *and* $\theta_1 = [(N-2)\alpha - 4]^+/(2(\alpha+2))$, $\theta_2 = [(N-2)\alpha - 2]^+/(2\alpha)$. *Then the problem* (3.1) *admits a unique global solution* $u \in \mathscr{W}_*$ *in the class* (2.23). *Moreover, this solution u satisfies*

$$\|u''(t)\|^2 + \|A^{1/2}u'(t)\|^2 + \|Au(t)\|^2 \leqslant Ce^{-kt} \tag{3.10}$$

with some constant $k > 0$.

Proof. Multiplying Eq. (3.1) by $2u'$ and integrating it over Ω, we have

$$E'(t) + 2\|u'(t)\|^2 = 0, \tag{3.11}$$

and hence, $E(t)$ is non-increasing. Integrating (3.11) over $[0, t]$, we get

$$E(t)+2\int_0^t \|u'(s)\|^2\,ds=E(0). \tag{3.12}$$

Since $u_0 \in \mathscr{W}_*$ and $\mathscr{W}_*$ is an open set, putting

$$T_5 \equiv \sup\{t \in [0, \infty): u(s) \in \mathscr{W}_* \text{ for } 0 \leqslant s < t\}$$

we see that $T_5 > 0$ and $u(t) \in \mathscr{W}_*$ for $0 \leqslant t < T_5$. If $T_5 < \infty$, then $u(T_5) \in \partial \mathscr{W}_*$, that is, one of the following two cases happens:

$$K(u(T_5))=0 \qquad \text{and} \qquad u(T_5) \neq 0, \tag{3.13}$$

or

$$K(u(t)) \to \infty \qquad \text{as} \qquad t \to T_5^-. \tag{3.14}$$

We see from (3.5)–(3.7) and (3.12) that

$$\|u(t)\|_{\alpha+2}^{\alpha+2} \leqslant (1/2)\, G_1(t)\, \|A^{1/2}u(t)\|^2 \tag{3.15}$$

for $0 \leqslant t < T_5$, where we set

$$G_1(t) \equiv c_3 E(0)^{(\alpha-(\alpha+2)\,\theta_1)/2}\, E_2(t)^{(\alpha+2)\,\theta_1/2} \tag{3.16}$$

with $c_3 = 2c_*^{\alpha+2}(1+2/\alpha)^{(\alpha-(\alpha+2)\,\theta_1)/2}$ under $\alpha-(\alpha+2)\,\theta_1>0$ (i.e., (3.8)).

Since $G_1(0)<1$ for small initial energy (i.e., (3.9)), putting

$$T_6 \equiv \sup\{t \in [0, \infty): G_1(s) < 1 \text{ for } 0 \leqslant s < t\}$$

we see that $T_6 > 0$ and $G_1(t) < 1$ for $0 \leqslant t < T_6$. If $T_6 < T_5$, then

$$G_1(T_6)=1, \tag{3.17}$$

and from (3.4) and (3.15) we have

$$K(u(t)) \geqslant \|A^{1/2}u(t)\|^2 - (1/2)\, G_1(t)\, \|A^{1/2}u(t)\|^2 \geqslant (1/2)\, \|A^{1/2}u(t)\|^2$$

for $0 \leqslant t \leqslant T_6$.

Next, we shall estimate the second energy $E_2(t)$ given by (3.5). Multiplying Eq. (3.1) by $2Au' + 2\varepsilon Au$, $0<\varepsilon<1$, and integrating it over Ω, we have that

$$\begin{aligned}
&\partial_t E_2^*(t) + 2(1-\varepsilon)\, \|A^{1/2}u'(t)\|^2 + 2\varepsilon(1+b\, \|A^{1/2}u(t)\|^{2\gamma})\, \|Au(t)\|^2 \\
&\quad = b(\partial_t\, \|A^{1/2}u(t)\|^{2\gamma})\, \|Au(t)\|^2 + 2(A^{1/2}f(u(t)),\, A^{1/2}u'(t)+\varepsilon A^{1/2}u(t)) \\
&\quad \equiv I_3(t)+I_4(t),
\end{aligned} \tag{3.18}$$

where we set $E_2^*(t) \equiv E_2(t) + 2\varepsilon(u'(t), Au(t)) + \varepsilon \|A^{1/2}u(t)\|^2$. Then, we see that

$$2^{-1}E_2(t) \leqslant E_2^*(t) \leqslant 2E_2(t) \tag{3.19}$$

for $\varepsilon = (2c_*)^{-1}$ and

$$\begin{aligned} &2(1-\varepsilon) \|A^{1/2}u'(t)\|^2 + 2\varepsilon(1 + b \|A^{1/2}u(t)\|^{2\gamma}) \\ &\qquad \geqslant c_*^{-1}E_2(t) \geqslant (2c_*)^{-1} E_2^*(t). \end{aligned} \tag{3.20}$$

To proceed the estimate, we observe that

$$I_3 \leqslant 2b\gamma \|A^{1/2}u\|^{2\gamma-1} \|A^{1/2}u'\| \|Au\|^2$$

and

$$\begin{aligned} I_4 &\leqslant c_*(\alpha+1) \|u\|_{N\alpha}^{\alpha} \|Au\| (2 \|A^{1/2}u'\| + c_*^{-1} \|A^{1/2}u\|) \\ &\leqslant c_*^{\alpha+1}(\alpha+1) \|A^{1/2}u\|^{\alpha(1-\theta_2)} \|Au\|^{\alpha\theta_2+1} (2 \|A^{1/2}u'\| + \|Au\|) \end{aligned}$$

with $\theta_2 = [(N-2)\alpha - 2]^+/(2\alpha)$ (<1), and hence, we get from (3.6) and (3.12) that

$$\begin{aligned} I_3(t) + I_4(t) &\leqslant 2b\gamma((1+2\alpha^{-1}) E(0))^{\gamma-1/2} E_2(t)^{3/2} \\ &\qquad + 2c_*^{\alpha+1}(\alpha+1)((1+2\alpha^{-1}) E(0))^{\alpha(1-\theta_2)/2} E_2(t)^{\alpha\theta_2/2+1} \\ &\leqslant (4c_*)^{-1} F_1(t) E_2^*(t), \end{aligned}$$

where we set

$$F_1(t) \equiv c_4 bE(0)^{\gamma-1/2} E_2(t)^{3/2} + c_5 E(0)^{\alpha(1-\theta_2)/2} E_2(t)^{\alpha\theta_2/2+1} \tag{3.21}$$

with $c_4 = 2^3 c_* \gamma(1+2/\alpha)^{\gamma-1/2}$ and $c_5 = 2^3 c_*^{\alpha+2}(\alpha+1)(1+2/\alpha)^{\alpha(1-\theta_2)/2}$. Therefore, it follows from (3.18) and (3.20) that

$$\partial_t E_2^*(t) + (2c_*)^{-1} [1-(1/2) F_1(t)] E_2^*(t) \leqslant 0. \tag{3.22}$$

Since $F_1(0) < 1$ for small initial energy (i.e., (3.9)), putting

$$T_7 \equiv \sup\{t \in [0, \infty): F_1(s) < 1 \text{ for } 0 \leqslant s < t\}$$

we see that $T_7 > 0$ and $F_1(t) < 1$ for $0 \leqslant t < T_7$. If $T_7 < T_6$, then

$$F_1(T_7) = 1. \tag{3.23}$$

Moreover, we see from (3.22) that

$$\partial_t E_2^*(t) \leqslant -(4c_*)^{-1} E_2^*(t)$$

and hence, from (3.19) that

$$E_2(t) \leqslant 4E_2(0) \exp\{-(4c_*)^{-1} t\} \tag{3.24}$$

for $0 \leqslant t \leqslant T_7$. Then, it follows from (3.9), (3.21), (3.24) that

$$F_1(t) \leqslant d_6 b E(0)^{\gamma - 1/2} E_2(0)^{1/2} + d_7 E(0)^{\alpha(1-\theta_2)/2} E_2(0)^{\alpha\theta_2/2} < 1 \tag{3.25}$$

for $0 \leqslant t \leqslant T_7$ with $d_6 = 2c_3$ and $d_7 = 2^{\alpha\theta_2} c_4$, which contradicts (3.23), and hence, we see $T_7 \geqslant T_6$.

Moreover, it follows from (3.9), (3.16), and (3.24) that

$$G_1(t) \leqslant d_5 E_1(0)^{(\alpha - (\alpha+2)\theta_1)/2} E_2(0)^{(\alpha+2)\theta_1/2} < 1$$

for $0 \leqslant t \leqslant T_6$ with $d_5 = 2^{(\alpha+2)\theta_1} c_3$, which contradicts (3.17), and hence, we see $T_6 \geqslant T_5$. By the similar argument as the proof of Theorem 2.4, we see that the both cases (3.13) and (3.14) do not happen, and hence, $T_5 = \infty$. Moreover, (3.24) holds true for $t \geqslant 0$ and the local solution u of Eq. (3.1) in the sense of Theorem 1.1 can be continued globally in time.

By the similarly way as in (2.43), we have that $\|u''(t)\| \leqslant Ce^{-kt}$ for $t \geqslant 0$. The proof of Theorem 3.2 is now completed. ∎

4. NON-POSITIVE INITIAL ENERGY CASE

In this section we consider the blowup problem for the following degenerate or non-degenerate wave equations with the blowup term $f(u) = |u|^\alpha u$,

$$\begin{cases} u'' + (a + b\,\|A^{1/2}u\|^{2\gamma})\, Au + \delta u' = |u|^\alpha u & \text{in} \quad \Omega \times [0, \infty) \\ u(0) = u_0, \qquad u'(0) = u_1, \qquad \text{and} & u|_{\partial\Omega} = 0, \end{cases} \tag{4.1}$$

where $a \geqslant 0$, $b \geqslant 0$, $a + b > 0$, $\gamma > 0$, $\delta \geqslant 0$, and $\alpha > 0$. We recall the energy associated with Eq. (4.1):

$$E(u, u') \equiv \|u'\|^2 + \left(a + \frac{b}{\gamma+1}\|A^{1/2}u\|^{2\gamma}\right)\|A^{1/2}u\|^2 - \frac{2}{\alpha+2}\|u\|_{\alpha+2}^{\alpha+2}. \tag{4.2}$$

Then we see that

$$E(t) + 2\delta \int_0^t \|u'(s)\|^2 \, ds = E(0) \tag{4.3}$$

for $t \geqslant 0$, where $E(t) \equiv E(u(t), u'(t))$ (see (2.12), (3.12)).

To show the blowup properties of the solutions, we implement the so-called concavity method (see Levine [10, 11] and also [30, 22]). We define a function $P(t)$ by

$$P(t) \equiv \|u(t)\|^2 + \delta \left\{ \int_0^t \|u(s)\|^2 \, ds + (T_0 - t) \, \|u_0\|^2 \right\} + r(t+\tau)^2 \quad (\geqslant 0) \tag{4.4}$$

for a solution $u(t)$, $t \in [0, T_0]$, where $T_0 > 0$, $r \geqslant 0$, and $\tau > 0$ are certain constants which are specified later on, then we observe the following property.

PROPOSITION 4.1. *The function $P(t)$ satisfies*

$$P(t)P''(t) - (\alpha/4 + 1) \, P'(t)^2 \geqslant P(t) \, Q(t), \tag{4.5}$$

where

$$Q(t) = -(\alpha + 2)(E(0) + r) + \left(a\alpha + b \, \frac{\alpha - 2\gamma}{\gamma + 1} \, \|A^{1/2}u(t)\|^{2\gamma} \right) \|A^{1/2}u(t)\|^2. \tag{4.6}$$

Proof. Differentiating (4.4) with respect to t, we have

$$\begin{aligned} P'(t) &= 2(u(t), u'(t)) + \delta\{ \|u(t)\|^2 - \|u_0\|^2 \} + 2r(t+\tau) \\ &= 2 \left\{ (u(t), u'(t)) + \delta \int_0^t (u(s), u'(s)) \, ds + r(t+\tau) \right\} \end{aligned}$$

and

$$\begin{aligned} P''(t) &= 2\{ (u(t), u''(t) + \delta u'(t)) + \|u'(t)\|^2 + r\} \\ &= 2\{ -(a + b \, \|A^{1/2}u(t)\|^{2\gamma}) \, \|A^{1/2}u(t)\|^2 \\ &\qquad + \|u(t)\|_{\alpha+2}^{\alpha+2} + \|u'(t)\|^2 + r\}, \end{aligned} \tag{4.7}$$

where we used Eq. (4.1). We set

$$
\begin{aligned}
R(t) \equiv & \left\{ \|u(t)\|^2 + \delta \int_0^t \|u(s)\|^2 \, ds + r(t+\tau)^2 \right\} \\
& \times \left\{ \|u'(t)\|^2 + \delta \int_0^t \|u'(s)\|^2 \, ds + r \right\} \\
& - \left\{ (u(t), u'(t)) + \delta \int_0^t (u(s), u'(s)) \, ds + r(t+\tau) \right\}^2,
\end{aligned}
$$

then $R(t) \geqslant 0$ and

$$
\begin{aligned}
R(t) = & \{ P(t) - \delta(T_0 - t) \|u_0\|^2 \} \\
& \times \left\{ \|u'(t)\|^2 + \delta \int_0^t \|u'(s)\|^2 \, ds + r \right\} - (1/4) P'(t)^2
\end{aligned}
$$

or

$$
\begin{aligned}
P'(t)^2 = 4 \Bigg[& \{ P(t) - \delta(T_0 - t) \|u_0\|^2 \} \\
& \times \left\{ \|u'(t)\|^2 + \delta \int_0^t \|u'(s)\|^2 \, ds + r \right\} - R(t) \Bigg]. \qquad (4.8)
\end{aligned}
$$

Thus, it follows from (4.8) that

$$
\begin{aligned}
& P(t) \, P''(t) - (\alpha/4 + 1) \, P'(t)^2 \\
& \geqslant P(t) \left[P''(t) - (\alpha + 4) \left\{ \|u'(t)\|^2 + \delta \int_0^t \|u'(s)\|^2 \, ds + r \right\} \right].
\end{aligned}
$$

Moreover, we observe from (4.7) and (4.3) that

$$
\begin{aligned}
& P''(t) - (\alpha + 4) \left\{ \|u'(t)\|^2 + \delta \int_0^t \|u'(s)\|^2 \, ds + r \right\} \\
& = -(\alpha + 2) \left\{ \|u'(t)\|^2 + 2\delta \int_0^t \|u'(s)\|^2 \, ds - \frac{2}{\alpha + 2} \|u(t)\|_{\alpha+2}^{\alpha+2} + r \right\} \\
& \quad - 2(a + b \|A^{1/2}u(t)\|^{2\gamma}) \|A^{1/2}u(t)\|^2 \\
& = -(\alpha + 2)(E(0) + r) + \left(a\alpha + b \frac{\alpha - 2\gamma}{\gamma + 1} \|A^{1/2}u(t)\|^{2\gamma} \right) \|A^{1/2}u(t)\|^2 \\
& = Q(t). \qquad (4.9)
\end{aligned}
$$

The proof of Proposition 4.1 is now completed. ∎

THEOREM 4.2. *Suppose that* $\alpha \geqslant 2\gamma$ *and that*

$$E(0) < 0, \qquad \text{or} \qquad E(0) = 0 \qquad \text{and} \qquad (u_0, u_1) > 0. \tag{4.10}$$

Then the local solution u of Eq. (4.1) *in the sense of Theorem* 1.1 *can not be continued to some finite time* $T > 0$. *Moreover, the finite time T is estimated such that*

$$T \leqslant \alpha^{-2}(-E(0))^{-1} \left[\{(2\delta \|u_0\|^2 - \alpha(u_0, u_1))^2 + \alpha^2(-E(0)) \|u_0\|^2\}^{1/2} + (2\delta \|u_0\|^2 - \alpha(u_0, u_1)) \right] \tag{4.11}$$

when $E(0) < 0$, *and*

$$T \leqslant 2\alpha^{-1}(u_0, u_1)^{-1} \|u_0\|^2 \tag{4.12}$$

when $E(0) = 0$ *and* $(u_0, u_1) > 0$.

Proof. We take $r = -E(0)$ ($\geqslant 0$) in (4.4), then we see from (4.6) that

$$Q(t) = \left(a\alpha + b \frac{\alpha - 2\gamma}{\gamma + 1} \|A^{1/2}u(t)\|^{2\gamma} \right) \|A^{1/2}u(t)\|^2 \geqslant 0$$

under $\alpha \geqslant 2\gamma$, and moreover, from (4.5) that

$$(P(t)^{-\alpha/4})'' = -(\alpha/4)\, P(t)^{-(\alpha/4+2)} \times \{P(t)\, P''(t) - (\alpha/4 + 1)\, P'(t)^2\} \leqslant 0. \tag{4.13}$$

Case 1. We assume that $E(0) < 0$. If we choose T_0 and τ such that

$$P'(0) > 0 \qquad \text{and} \qquad (0 <)\, 4P(0)/(\alpha P'(0)) \leqslant T_0$$

then we have from (4.13) that

$$P(t) \geqslant \left\{ \frac{4P(0)^{\alpha/4+1}}{4P(0) - \alpha P'(0)t} \right\}^{4/\alpha}$$

for some $t > 0$, and hence, there exists a T such that

$$T \leqslant T_0 \qquad \text{and} \qquad \lim_{t \to T^-} \left\{ \|u(t)\|^2 + \delta \int_0^t \|u(s)\|^2\, ds \right\} = \infty, \tag{4.14}$$

that is, $\lim_{t \to T^-} \|u(t)\|^2 = \infty$.

Next, we shall show (4.11). If τ is sufficiently large, we see

$$P'(0) = 2\{(u_0, u_1) + (-E(0))\, \tau\} > 0$$

and

$$4P(0)/(\alpha P'(0)) \leqslant T_0. \tag{4.15}$$

We note that (4.15) if and only if

$$T(\tau) \equiv \frac{2(\|u_0\|^2 + (-E(0))\,\tau^2)}{\alpha\{(u_0, u_1) + (-E(0))\,\tau\} - 2\delta\,\|u_0\|^2} \leqslant T_0$$

We find that $T(\tau)$ take a minimum on $(0, \infty)$ at

$$\tau = \tau_0 \equiv \alpha^{-2}(-E(0))^{-1}\,[\{(2\delta\,\|u_0\|^2 - \alpha(u_0, u_1))^2 + \alpha^2(-E(0))\,\|u_0\|^2\}^{1/2} + (2\delta\,\|u_0\|^2 - \alpha(u_0, u_1))].$$

Thus, if we put $T_0 = T(\tau_0)$, we arrive at (4.11).

Case 2. We assume that $E(0) = 0$ and $(u_0, u_1) > 0$. Then we see

$$P'(0) = 2(u_0, u_1) > 0 \qquad \text{and} \qquad P(0) = \|u_0\|^2 > 0$$

Thus, (4.14) with $T_0 = 4P(0)/(\alpha P'(0))$ follows from (4.13), and we get (4.12). The proof of Theorem 4.2 is now completed. ∎

5. POSITIVE INITIAL ENERGY CASE

We have already proved that the local solution of (4.1) blows up under $E(0) \leqslant 0$ in previous section. In this section, when $\alpha \leqslant 4/(N-2)$ ($\alpha < \infty$ if $N = 1, 2$), even if the initial energy $E(0)$ is positive, we shall show that the local solution blows up under

$$u_0 \in \mathscr{V}_* \equiv \{u \in \mathscr{D}(A) : K(u) < 0\} \qquad \text{and} \qquad E(0) < D_a, \tag{5.1}$$

where

$$K(u) \equiv \begin{cases} a\,\|A^{1/2}u\|^2 - \|u\|_{\alpha+2}^{\alpha+2} & \text{if} \quad a > 0 \\ b\,\|A^{1/2}u\|^{2(\gamma+1)} - \|u\|_{\alpha+2}^{\alpha+2} & \text{if} \quad a = 0 \end{cases} \tag{5.2}$$

and

$$D_a \equiv \begin{cases} \dfrac{\alpha}{\alpha+2}\left(\dfrac{a}{c_*^2}\right)^{(\alpha+2)/\alpha} & \text{if} \quad a > 0 \\ \dfrac{\alpha - 2\gamma}{(\alpha+2)(\gamma+1)}\left(\dfrac{b}{c_*^{2(\gamma+1)}}\right)^{(\alpha+2)/(\alpha-2\gamma)} & \text{if} \quad a = 0 \end{cases} \tag{5.3}$$

with the Sobolev–Poincaré's best constant

$$c_* \equiv \sup\{ \|v\|_{\alpha+2}/\|A^{1/2}v\| : v \in \mathscr{D}(A^{1/2}),\ v \neq 0\} \qquad (>0) \tag{5.4}$$

for $\alpha \leqslant 4/(N-2)$ ($\alpha < \infty$ if $N = 1, 2$).

We observe the following useful result related to the K-negative set $\mathscr{V}_*$.

PROPOSITION 5.1. *Let $u(t)$ be a solution of Eq.* (4.1), *and let*

$$u_0 \in \mathscr{V}_* \qquad \text{and} \qquad E(0) < D_a. \tag{5.5}$$

Suppose that

$$\alpha \leqslant 4/(N-2) \qquad (\alpha < \infty \ \text{if}\ N = 1, 2)$$

and

$$\alpha \geqslant 2\gamma \quad \text{if} \quad a > 0, \qquad \text{or} \qquad \alpha > 2\gamma \quad \text{if} \quad a = 0.$$

Then

$$K(u(t)) < 0 \qquad \text{and} \qquad u(t) \neq 0 \tag{5.6}$$

and

$$E(t) < D_a \leqslant \frac{a\alpha}{\alpha+2} \|A^{1/2}u(t)\|^2 + \frac{b(\alpha-2\gamma)}{(\alpha+2)(\gamma+1)} \|A^{1/2}u(t)\|^{2(\gamma+1)}. \tag{5.7}$$

Proof. Since $E(t) \leqslant E(0)$ by (4.3), we get from (5.5) immediately that

$$E(t) < D_a. \tag{5.8}$$

Since $u_0 \in \mathscr{V}_*$ and $\mathscr{V}_*$ is an open set, putting

$$T \equiv \sup\{ t \in [0, \infty) : K(u(s)) < 0 \text{ for } 0 \leqslant s < t\},$$

we see that $T > 0$ and $K(u(t)) < 0$ and $u(t) \neq 0$ for $0 \leqslant t < T$. If $T < \infty$, then $K(u(T)) = 0$ and

$$J(u(T)) \equiv \left(a + \frac{b}{\gamma+1} \|A^{1/2}u(T)\|^{2\gamma}\right) \|A^{1/2}u(T)\|^2 - \frac{2}{\alpha+2} \|u(T)\|_{\alpha+2}^{\alpha+2}$$

$$\geqslant \begin{cases} \dfrac{a\alpha}{\alpha+2} \|A^{1/2}u(T)\|^2 & \text{if} \quad a > 0 \\[2ex] \dfrac{b(\alpha-2\gamma)}{(\alpha+2)(\gamma+1)} \|A^{1/2}u(T)\|^{2(\gamma+1)} & \text{if} \quad a = 0. \end{cases} \tag{5.9}$$

Now, when $K(u)>0$ and $u\neq 0$, we see from (5.4) that

$$c_*^{\alpha+2}\,\|A^{1/2}u\|^{\alpha+2}\geqslant \|u\|_{\alpha+2}^{\alpha+2}>\begin{cases} a\,\|A^{1/2}u\|^2 & \text{if}\quad a>0\\ b\,\|A^{1/2}u\|^{2(\gamma+1)} & \text{if}\quad a=0\end{cases}$$

under $\alpha\leqslant 4/(N-2)$ ($\alpha<\infty$ if $N=1,\,2$), and hence,

$$\|A^{1/2}u\|>\begin{cases}(a/c_*^{\alpha+2})^{1/\alpha} & \text{if}\quad a>0\\ (b/c_*^{\alpha+2})^{1/(\alpha-2\gamma)} & \text{if}\quad a=0.\end{cases}\tag{5.10}$$

Thus, we have from (5.10) and the continuity that

$$\|A^{1/2}u(T)\|\geqslant\begin{cases}(a/c_*^{\alpha+2})^{1/\alpha} & \text{if}\quad a>0\\ (b/c_*^{\alpha+2})^{1/(\alpha-2\gamma)} & \text{if}\quad a=0.\end{cases}\tag{5.11}$$

Therefore, it follows from (5.9) and (5.11) that

$$E(T)\geqslant J(u(T))\geqslant D_a,$$

which contradicts (5.8), and hence, we see $T=\infty$ and we get (5.6). Moreover, from (5.8) and (5.10), we obtain (5.7). ∎

Our result is as follows. (Cf. [23, 8] for $\delta=0$ and [19] for $\delta\geqslant 0$.)

THEOREM 5.2. *Let the initial data* $\{u_0, u_1\}$ *belong to* $\mathscr{V}_*\times\mathscr{D}(A^{1/2})$ *and satisfy*

$$E(0)<D_a\tag{5.12}$$

with the positive constant D_a *given by* (5.3). *Suppose that*

$$\alpha\leqslant 4/(N-2)\qquad(\alpha<\infty\ \textit{if}\ N=1,\,2)$$

and

$$\alpha\geqslant 2\gamma\quad \textit{if}\quad a>0,\qquad \textit{or}\qquad \alpha>2\gamma\quad\textit{if}\quad a=0.$$

Then the local solution u *of* (4.1) *in the sense of Theorem* 1.1 *can not be continued to some finite time* $T>0$. *Moreover, if* $(u_0, u_1)>0$, *the finite time* T *is estimated such that*

$$T\leqslant 2\alpha^{-1}(u_0,u_1)^{-1}\,\|u_0\|^2.\tag{5.13}$$

Proof. We take $r=0$ in (4.4), then we see from (4.6), (5.7), (5.12) that

$$Q(t)=(\alpha+2)\left\{-E(0)+\frac{a\alpha}{\alpha+2}\|A^{1/2}u(t)\|^2+\frac{b(\alpha-2\gamma)}{(\alpha+2)(\gamma+1)}\|A^{1/2}u(t)\|^{2(\gamma+1)}\right\}$$

$$\geqslant(\alpha+2)\{-E(0)+D_a\}\equiv C_1>0. \tag{5.14}$$

Moreover, we observe from (4.9) with $r=0$ and (5.14) that

$$P''(t)\geqslant Q(t)\geqslant C_1.$$

Then we obtain

$$P'(t)\geqslant P'(0)+C_1 t=2(u_0,u_1)+C_1 t,$$

and hence, there exists a $t_0\geqslant 0$ such that

$$P'(t)>0 \qquad \text{for} \quad t\geqslant t_0,$$

that is, $P(t_0)>0$ and $P'(t_0)>0$. We note that if $(u_0,u_1)>0$, we can take $t_0=0$. Thus, we observe from (4.5) and (5.14) that

$$(P(t)^{-\alpha/4})''=-(\alpha/4)\,P(t)^{-(\alpha/4+2)}\{P(t)\,P''(t)-(\alpha/4+1)\,P'(t)^2\}\leqslant 0$$

and

$$P(t)\geqslant\left\{\frac{4P(t_0)^{\alpha/4+1}}{4P(t_0)-\alpha P'(t_0)}\right\}^{\alpha/4}$$

for some $t>t_0$, and hence, (4.14) with $T_0=4P(t_0)/(\alpha P'(t_0))$ follows and we get (5.13) if $(u_0,u_1)>0$ (i.e. $t_0=0$). The proof of Theorem 5.2 is now completed. ▮

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